ANALYSIS OF LYAPUNOV STABILITY THEORY FOR DYNAMICAL SYSTEMS. MATHCAD VERIFICATION ALGORITHMS

Eng. Dan Mihai MIRONESCU*
M&M COMPUTERS, Ploieşti

This article contains a synthesis of the basic concepts from the Lyapunov's stability theory defining the notions of equilibrium, uniform stability, asymptotic stability, globally exponential stability and Lyapunov functions.

Lyapunov's direct method is discussed, complete with an analysis of the evolution of a dynamical system in the phase plane conducted with the aid of Mathcad software.

For dynamical autonomous systems the LaSalle theorem is used in the analysis of stability of motion, complementary to Lyapunov's direct method.

Keywords: Lyapunov stability; Mathcad Programme; stability analysis of manipulators and robot motion.

Lyapunov's stability theory, developed in the last decade of the XIXth century, is widely used in various physics and technical domains.

In 1946 N. G. Cetaev published the work "Stability of Motion" and in 1966 the monograph work "Stability of Motion Theory" of I. G. Malkin appears.

In this article algorithms for verifying the stability of motion are utilized.

Lyapunov Stability

The main objective in Lyapunov's stability theory is to study the behavior of dynamical systems described by ordinary differential equations as the following:

$$\frac{dx}{dt} = f(x,t), \quad x \in R^n, \quad t \in R_+ \tag{1}$$

in which the vector *x* corresponds to the position of the dynamical system in question at the moment *t* with the initial condition $x(t_0)$, for $t_0 \geq 0$.

* e-mail: *dan@mmc.ro*

For simplification purposes $x(t)$ will be used to denote a solution for the differential equation (1) instead of $x(t, t_0, x(t_0))$.

Function $f(x,t)$ is assumed to be continuous in *t* and *x* and is in such way that equation (1) has an unique solution for each initial condition.

Generally it is assumed that a solution for $t \geq t_0 \geq 0$ exists or, if restrictions exist, it can be assumed that a solution exists on a finite interval. Worth emphasizing is that, if found on an infinite interval, this can be solved with Lyapunov stability theory.

Further we assume that a finite interval exists. If function f is not explicitly time dependent, equation (1) becomes:

$$\frac{dx}{dt} = f(x), \quad x \in R^n \qquad (2)$$

and it is said that the system is autonomous.

In this case a discussion about the initial moment t_0 has no meaning however it is considered that, for t_{01} and t_{02}, given with $x(t_{01}) = x(t_{02})$, we have:

$x(t_{91} + T, t_{01}, x(t_{01})) = x(t_{02} + T, t_{02}, x(t_{02}))$, for any $T \geq 0$.

As a consequence for all autonomous differential equations we may consider that $t_0 = 0$.

If $f(x,t) = A(t) \cdot x + u(t)$ with $A(t)$ is a quadratic matrix of *n* size, $A(t)$ and vector *u(t)* are only functions of *t* or constant then the differential equation (1) is defined as linear, conversely being defined as nonlinear.

Lyapunov's Theory Concepts

Among the basic concepts in Lyapunov's theory we can mention equilibrium, stability, asymptotic stability, homogenous exponential stability.

Equilibrium Definition

The constant vector $x_e \in R^n$ represents a state of equilibrium – or is equilibrium – of system (1) if:

$$f(t, x_e) = 0, \ \forall \ t \geq 0. \qquad (3)$$

A direct consequence of this definition is that, if the initial state $x(t_0) \in R^n$ is an equilibrium $x(t_0) = x_e \in R^n$, then:

$$x(t) = x_e, \ \forall \ t \geq t_0 \geq 0,$$

$$\frac{d}{dt}(x(t)) = 0, \ \forall \ t \geq t_0 \geq 0.$$

Usually it is assumed that the origin of the space R^n state is $x = 0 \in R^n$, being a condition of equilibrium of system (1). If this statement is not true then it can be shown, by means of an appropriate variable change, that any condition of equilibrium of system (1) can be transferred to origin.

Generally a differential equation can have more than one equilibrium, this number can even be infinite, nevertheless it is also possible that no equilibrium exists for a given differential equation.

Application. The following differential equation is given:

$$\frac{dx(t)}{dt} = a \cdot x + b \cdot u(t),$$

with initial conditions $(t_0, x(t_0)) \in R_+ \times R$, where $a \neq 0$ and $b \neq 0$ are real constants and $u : R_+ \to R$ is a continuous function.

If $u(t) = u_0$ for all values t ≥ 0, and the differential equation is autonomous, then the only point of equilibrium of the equation is: $x_e = -\frac{b}{a} \cdot x_e = -\frac{b \cdot u_0}{a}$.

To be noted that the autonomous nonlinear system: $\frac{dx(t)}{dt} = \exp(-x)$ has no point of equilibrium.

The following autonomous nonlinear differential equations system is considered:

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = \sin(x_1).$$

This system has an infinite number of isolated equilibrium positions given by:

$$x_e = [x_{1e} \quad x_{2e}]^T = [n \cdot \pi \quad 0]^T, \text{ cu } n = \ldots, -1, 0, 1, \ldots$$

Systems with multiple equilibrium positions are pretty frequent in practice, for instance the mechanisms and dynamic models of robot manipulators.

Without loss of generality we can assume that the origin of space $x = 0 \in R^n$ is an equilibrium for equation (1), therefore we would provide the definitions for stability from origin, but they can be reformulated for other equilibrium positions by means of appropriate modifications of the coordinates.

Stability

The origin is in a stable equilibrium – in the sense of Lyapunov – for the differential equation (1) if for any pair of numbers $\varepsilon > 0$ şi $t_0 \geq 0$ there is $\delta = \delta(t_0, \varepsilon) > 0$ so that:

$$\|x(t_0)\| < \delta \text{ implies } \|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0 \geq 0, \tag{4}$$

where $\|x\|$ is the euclidean norm of the vector $x \in R^n$ defined as:

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T \cdot x}.$$

Uniform Stability

Origin $x = 0 \in R^n$ is a variant of uniformly stable equilibrium – in the sense of Lyapunov – for equation (1) if for each number $\varepsilon > 0$ there is δ = δ (ε) > 0 so that equation (4) is fulfilled. The origin is uniformly stable if δ can be chosen independently from initial moment t_0. For autonomous systems uniform stability and equilibrium stability are equivalent notions.

Asymptotic Stability

The origin is an asymptotically stable equilibrium for differential equation (1) if:

- The origin is stable;
- The origin is attractive, meaning that for each $t_0 \geq 0$ there is $\delta' = \delta'(t_0) > 0$ such that: $\|x(t_0)\| < \delta'$ implies $\|x(t)\| \to 0$ for $t \to \infty$. (9)

The asymptotic stability for autonomous systems origin is valid if in the previous statement it is stated "there exists $\delta' > 0$."

Globally Exponential Stability

The origin of the differential equations system:

$$\frac{dx}{dt} = f(x,t), \quad x \in R^n, \quad t \in R_+$$

is a stable equilibrium at globally exponential level, if the constants α and β exist, independent of t_0, such that:

$$\|x(t)\| < \alpha \cdot \|x(t_0)\| \cdot \exp(-\beta \cdot (t - t_0)), \quad \forall\, t \geq t_0 \geq 0, \quad \forall\, x(t_0) \in R^n \tag{5}$$

Lyapunov Functions

This section contains definitions which determine a certain class of functions, considered as being essential in the usage of Lyapunov's direct method for studying the equilibrium stability of differential equations.

Locally And Globally Positive Definite Functions

A continuous function $W : R^n \to R_+$ is locally positive definite if:

1. $W(0) = 0;$
2. $W(x) > 0$ for small values $\|x\| \neq 0.$ (6)

A continuous function $W : R^n \to R_+$ is globally positive definite if:

1. $W(0) = 0;$
2. $W(x) > 0 \ \ \forall\ x \neq 0.$ (7)

For *n* continuous functions $V : R_+ \times R^n \to R_+$ which are time dependent, $V(t,x)$ is positive definite and locally positive definite respectively if:

1. $V(t,0) = 0, \ \forall\ t \geq 0;$
2. $V(t,x) \geq W(x), \ \forall\ t \geq 0, \forall\ x \in R^n,$ for small values of $\|x\|$

where *W(x)* is a positive definite function, locally positive definite respectively.

Limitless Radial Functions

A continuous function $W : R^n \to R$ is defined to be limitless radial if:

$W(x) \to \infty$ if $\|x\| \to \infty$,

$W(x) \to \infty$ for $x \to \infty$.

Lyapunov's Auxiliary Function

A continuous and differentiable function $V : R_+ \times R^n \to R_+$ is defined to be a Lyapunov auxiliary function for equilibrium $x = 0 \in R^n$ of the differential equation system:

$$\frac{dx}{dt} = f(t,x)$$

if the following conditions are met:

1. $V(t,x)$ is locally positive definite;
2. $\dfrac{\partial V(t,x)}{\partial t}$ is continuous with respect to *t* and *x*; (8)
3. $\dfrac{\partial V(t,x)}{\partial x}$ is continuous with respect to *t* and *x*.

Time Derivative Of Lyapunov's Auxiliary Function

Let $V(t,x)$ a Lyapunov auxiliary function for the system $\frac{dx}{dt}=f(t,x)$.

The total time derivative of the function $V(t,x)$ along the trajectories of the differential equation system is:

$$\frac{d}{dt}\left(V(t,x)\right)=\frac{\partial V(t,x)}{\partial t}+\frac{\partial V(t,x)}{\partial x}^{T}\cdot f(t,x). \quad (9)$$

The Lyapunov auxiliary function $V(t,x)$ for the system $\frac{dx}{dt}=f(t,x)$ is a Lyapunov function – for the system in question – if the total derivative fulfills the condition:

$$\frac{d}{dt}\left(V(t,x)\right)=\frac{\partial V(t,x)}{\partial t}+\frac{\partial V(t,x)}{\partial x}^{T}\cdot f(t,x)\le 0,\ \ \forall\ t\ \ge 0, \quad (10)$$

for small values of $\|x\|$.

Lyapunov's Direct Method

The origin is a stable equilibrium for the differential equation system:

$$\frac{dx}{dt}=f(t,x)$$

if a $V(t,x)$ Lyapunov auxiliary function exists, such that:

$$\frac{d}{dt}\left(V(t,x)\right)=\frac{\partial V(t,x)}{\partial t}+\frac{\partial V(t,x)}{\partial x}^{T}\cdot f(t,x)\le 0,\ \ \forall\ t\ \ge 0$$

for small values of $\|x\|$.

The LaSalle Theorem

Being the autonomous differential equation system $\frac{dx}{dt}=f(x)$ with the origin $x=0\in R^{n}$ in an equilibrium state. It is assumed that a Lyapunov auxiliary function exists, globally positive definite and limitless radial $V(x)$, such that:

$$\frac{dV}{dx}\le 0,\ \forall x\in R^{n}. \quad (11)$$

The following set is defined:

$$\Omega=\left\{x\in R^{n}:\frac{dV}{dx}=0\right\}. \quad (12)$$

If $x(0)=0$ is a state only in Ω and the solution $x(t)$ remains in Ω, which means $x(t)\in\Omega$ for all $t\geq 0$, then the origin $x=0\in R^n$ is globally asymptotically stable.

To be noted that it is not required, when applying the LaSalle theorem to determine asymptotic stability, for V(x) to be a negative definite function. It is to be reminded that this theorem can be utilized only for autonomous differential equations.

LaSalle Theorem Application For Two Differential Equation Systems

Being the following autonomous differential equation systems:

$$\frac{dx}{dt}=\frac{\partial f(x,z)}{\partial x},\quad x\in R^n \tag{13}$$

$$\frac{dz}{dt}=\frac{\partial f(x,z)}{\partial z},\quad z\in R^m \tag{14}$$

where $\frac{\partial f}{\partial x}(0,0)=0$ şi $\frac{\partial f}{\partial z}(0,0)=0$, which means that the origin is a point of equilibrium.

Being $V:R^n\times R^m\to R_+$ a globally positive definite function and radially non-limited in both arguments.

It is assumed that a globally positive definite function $W:R^m\to R_+$ exists such that:

$$\frac{dV}{dt}(x,z)=-W(z). \tag{15}$$

If *x=0* is an unique solution for $\frac{\partial f}{\partial z}(x,0)=0$ then the origin $\left[x^T z^T\right]^I=0$ is globally asymptotically stable.

Stability Analysis For Linear Harmonic Oscillator Motion

Being the linearly damped harmonic oscillator as shown in figure 1. The system dynamics is given by the differential equation:

$$m\cdot\frac{d^2q}{dt^2}+b\cdot\frac{dq}{dt}+k\cdot q=0, \tag{16}$$

where *m*, *b* and *k* are positive variables.

Phase Space Analysis

In the phase space the oscillator equation is expressed as:

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -\left(\frac{k}{m}\right)\cdot q - \left(\frac{b}{m}\right)\cdot \dot{q} \end{bmatrix} \qquad (17)$$

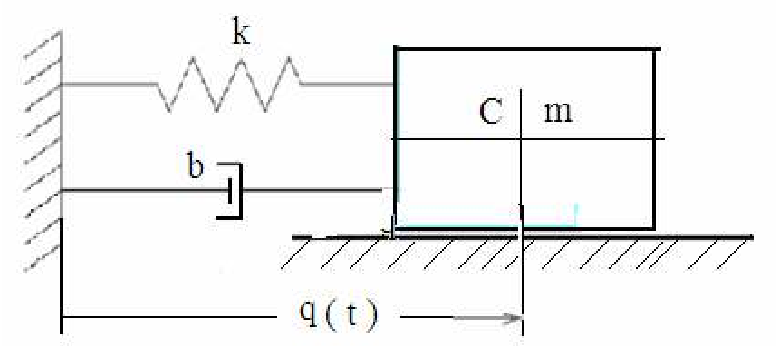

Fig. 1 *Linearly damped harmonic oscillator*

Numerical solving of the linear harmonic oscillator differential equation (16)

The following Mathcad algorithm is applied:

$m = 1\ [kg]$, $k = 0.02\ \left[\frac{N}{m}\right]$, $b = 0.02\left[\frac{N\cdot s}{m}\right]$, $T = 200\ [s]$

Given

$$m\cdot\left(\frac{d^2}{dt^2}q(t)\right) + b\cdot\frac{d}{dt}q(t) + k\cdot q(t) = 0$$

$q(0) = 0\ [m]$, $q'(0) = 0.01\ \left[\frac{m}{s}\right]$

The following graphs are generated:

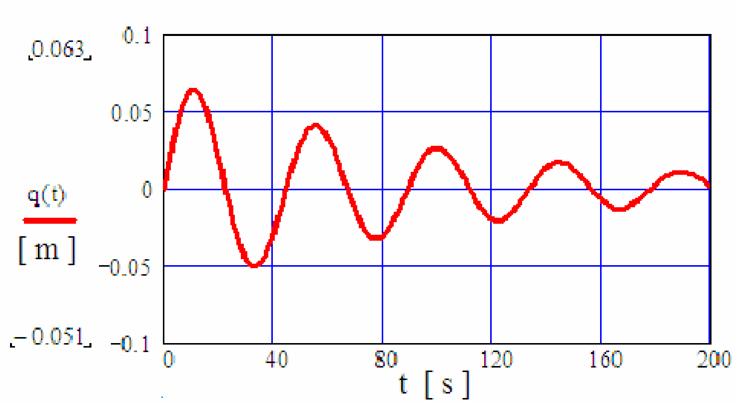

Fig. 2 *Variation with time of q*

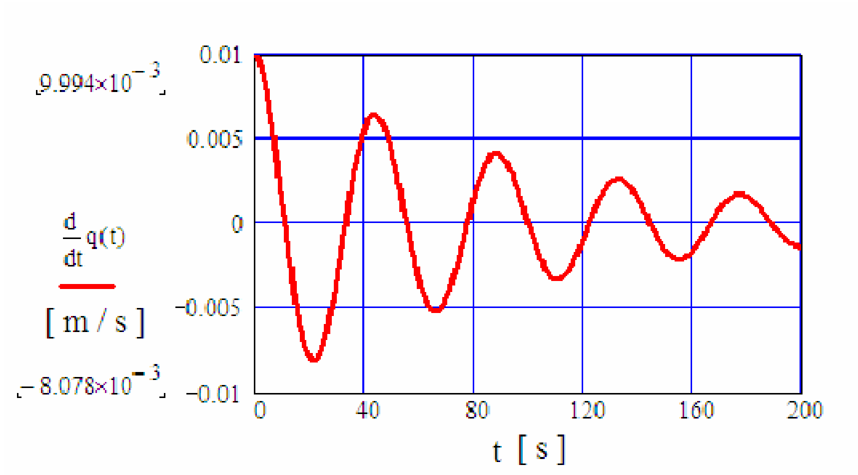

Fig. 3 *Variation of derivative for q(t) with time*

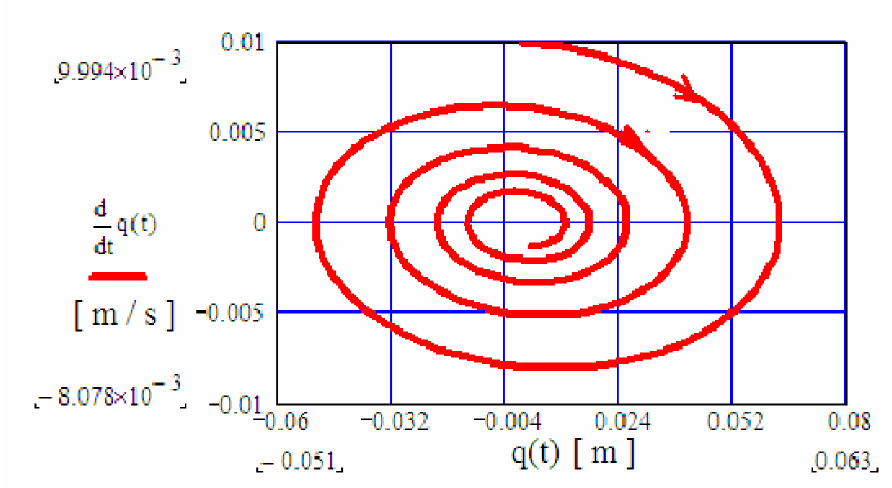

Fig. 4 *Variation of derivative for* $q(t)$ *in relation with* $q(t)$

The system state is defined with $x = (q, \dot{q})$.

Because this system is linear we can determine its stability by examining the poles of the system. The jacobian matrix for this system is:

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix},$$

which has the characteristic equation:

$$\lambda^2 + \left(\frac{b}{m}\right) \cdot \lambda + \left(\frac{k}{m}\right) = 0.$$

The roots of the equation are:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4km}}{2 \cdot m}$$

They always have a negative real part, therefore the system is globally exponentially stable.

Lyapunov's Direct Method Application

When applying Lyapunov's direct method – in order to determine exponential stability – quantification of the system energy is utilized as a function:

$$V(x,t)=\frac{1}{2}\cdot m\cdot\dot{q}^2+\frac{1}{2}\cdot k\cdot q^2.$$

After derivation with respect to time we obtain:

$$\dot{V}(x,t)=m\cdot\dot{q}\cdot\ddot{q}+k\cdot q\cdot\dot{q}=-b\cdot\dot{q}^2$$

$-\dot{V}(x,t)$ is a quadratic function but not locally positive because it is not dependent to *q* and therefore we cannot conclude the exponential stability analysis.

It is possible to conclude the asymptotic stability analysis by utilizing the LaSalle theorem yet, knowing that the system is conservative, it is also exponentially stable.

Conclusions

In order to solve a higher order differential equation in Mathcad the following algorithm is applied:

Given

$$a_0\cdot\frac{d^2}{dt^2}(x(t))+a_1\cdot\frac{d}{dt}(x(t))+a_2\cdot x(t)=f(t),$$

$$x(0)=x_0,\quad x'(0)=x_1,$$

$$x=\mathrm{Odesolve}(t,T_0).$$

The higher order ordinary differential equation can be transformed into a first order differential equation system.

By solving the following autonomous system in Mathcad:

$$\frac{dr}{dt}=a\cdot r\cdot(1-r),\quad \frac{d\theta}{dt}=\sin^2\left(\frac{\theta}{2}\right),\quad \theta\in[0,\ 2\pi]. \tag{18}$$

For $a=0.05$, $T=500\ [s]$, with initial conditions $r(0)=0.25\ [m]$, $\theta(0)\ [rad]$ we obtain:

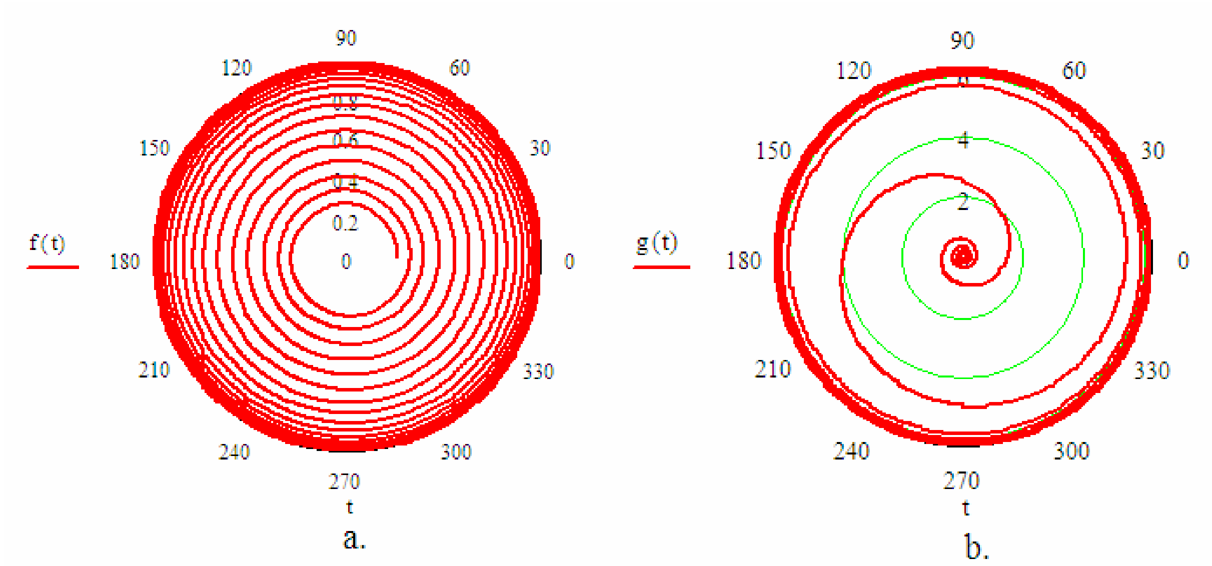

Fig. 5 *Variation of the radius vector in polar coordinates (a) and variation of the radius vector angular position (b) with respect to time*

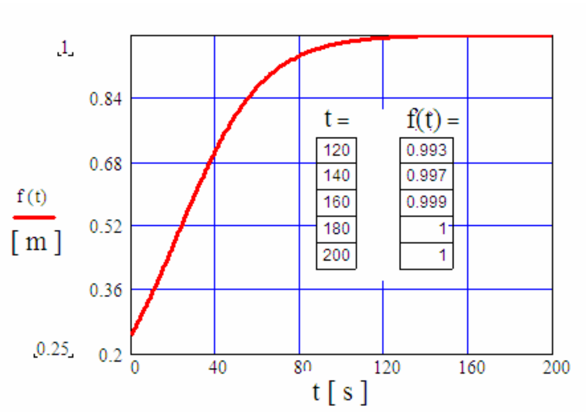

Fig. 6 *Variation with time of the radius vector in cartesian coordinates*

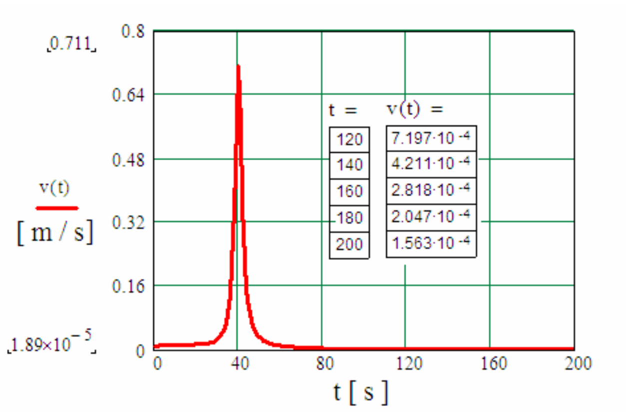

Fig. 7 *Variation with time of speed*

In figure 6 it can be observed that the dynamic system motion – defined by the differential equation system – is asymptotically stable.

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